SOLUTIONS OF EXERCISES.

64

A FLY-WHEEL weighing 12000 pounds with a rim of 10 feet diameter is driven by a piston pressure of 3000 pounds acting through a stroke of 3 feet. If the wheel be treated as concentrated in its rim, the weight of all other parts of the mechanism being neglected and the connecting-rod being treated as infinite in length, how many turns will the wheel make in 10 seconds?

[*De Volson Wood.*]

SOLUTION.

Let $\vartheta =$ angle between crank and connecting-rod,
$a =$ length of crank,
$F =$ piston pressure,
$I =$ moment of inertia of wheel.

Then
$$\frac{d^2\vartheta}{dt^2} = m \sin\vartheta, \tag{1}$$
where
$$m = Fa \,/\, I.$$

For the first stroke, let $a =$ initial value of ϑ. Then
$$\left(\frac{d\vartheta}{dt}\right)^2 = 2m(\cos a - \cos\vartheta), \tag{2}$$
$$\sqrt{(2m)}\,.\,t = \int_a^\theta \frac{d\vartheta}{\sqrt{(\cos a - \cos\vartheta)}}.$$

Let $\sin\varphi = \cos\frac{1}{2}\vartheta \,/\, \cos\frac{1}{2}a$. Then
$$\sqrt{(m)}\,.\,t = \int_\phi^{\frac{1}{2}\pi} \frac{d\varphi}{\Delta\varphi} = K - F(k,\varphi), \tag{3}$$
where $k = \cos\frac{1}{2}a$.

For any succeeding stroke, let $\omega =$ initial angular velocity, and $0 =$ initial value of ϑ; then
$$\left(\frac{d\vartheta}{dt}\right)^2 = \omega^2 + 2m(1 - \cos\vartheta). \tag{4}$$

Let $\vartheta = \pi - 2\varphi$; then
$$\tfrac{1}{2}\sqrt{(\omega^2 + 4m)}\,.\,t = \int_\phi^{\frac{1}{2}\pi} \frac{d\varphi}{\Delta\varphi} = K - F(k,\varphi), \tag{5}$$

where $k = \sqrt{\dfrac{4m}{\omega^2 + 4m}}$.

For the whole time of any stroke, $F(k, \varphi)$ in (3) or (5) becomes 0, and we have

$$h_n t_n = K_n, \tag{6}$$

the subscript denoting the number of the stroke; or

$$\log t_n = \log K_n - \log h_n. \tag{7}$$

Here $\quad h_1 = \sqrt{m}, \quad h_2 = \frac{1}{2}\sqrt{(\omega_2^2 + 4m)}, \quad h_3 = \frac{1}{2}\sqrt{(\omega_3^2 + 4m)}$, etc.

$$k_1 = \cos\frac{a}{2}, \quad k_2 = \sqrt{\frac{4m}{\omega_2^2 + 4m}}, \quad k_3 = \sqrt{\frac{4m}{\omega_3^2 + 4m}}, \text{ etc.}$$

From (2) and (4), $\omega_2^2 = 2m(\cos a + 1)$, $\omega_3^2 = 2m(\cos a + 3)$, etc. Assume $a = \frac{1}{2}\pi$, then $\omega_2^2 = 2m$, $\omega_3^2 = 6m$, $\omega_4^2 = 10m$, etc.

In the problem given, $F = 3000$, $a = \frac{3}{2}$, $I = \frac{12000}{32 \cdot 2} \times 25$; $\therefore m = 0.483$. Hence we may find

$\log h_1 = 9.8420$	$\log k_1 = 9.8495$	$\log K_1 = 0.2681$
$\log h_2 = 9.9300$	$\log k_2 = 9.9120$	$\log K_2 = 0.3073$
$\log h_3 = 0.0409$	$\log k_3 = 9.8010$	$\log K_3 = 0.2498$
$\log h_4 = 0.1140$	$\log k_4 = 9.7280$	$\log K_4 = 0.2319$
$\log h_5 = 0.1686$	$\log k_5 = 9.6734$	$\log K_5 = 0.2230$
$\log h_6 = 0.2122$	$\log k_6 = 9.6298$	$\log K_6 = 0.2177$

The values of $K_1, \ldots K_6$ were found from the table, Hoüel's *Calcul Infinitésimal*, Vol. IV, p. 302.

Substituting in (7), we may find

$$t_1 = 2.668, \quad t_2 = 2.384, \quad t_3 = 1.618, \quad t_4 = 1.312, \quad t_5 = 1.133.$$

The whole time of the first five strokes is 9.115 seconds.

To find the angle described during the remaining 0.885 seconds we have from (5),

$$\begin{aligned}\varphi &= \operatorname{am}\left[K_6 - \tfrac{1}{2}\sqrt{(\omega_6^2 + 4m)}\,.\,t_6\right], \quad \text{mod. } k_6\\ &= \operatorname{am} 0.208, \quad \text{mod. } 0.4264.\end{aligned}$$

By interpolation in the table (*Calcul Infinitésimal*, Vol. IV, p. 303), we find $\varphi = 0.132\pi/2$; $\therefore \vartheta = \pi - 2\varphi = 0.868\pi$. The whole angle described in 10 seconds is therefore $\frac{1}{2}\pi + 4\pi + 0.868\pi = 5.368\pi$, and the number of revolutions is 2.684. [*L. M. Hoskins.*]

66

SHOW that the difference between the radii of the inscribed and circumscribed circles of a triangle is a minimum when the triangle is equilateral.

[*R. D. Bohannan.*]

SOLUTION.

If r and R be the radii of the inscribed and circumscribed circles of the triangle ABC, considering R as constant, we shall have

$$1 - r/R = 2 - \cos A - \cos B - \cos C = \text{min.}$$

Considering C as constant,

$$dA = -dB,$$
$$-\sin A + \sin B = 0,$$

and similarly with regard to B, C and C, A;

$$\therefore A = B = C. \qquad [\textit{Ormond Stone.}]$$

67

A WINS from B on an average 8 games of chess out of 11. What is the value of A's chance in a match for $100 to be given to the player who shall first score three games; drawn games not to be counted? [*L. G. Barbour.*]

SOLUTION.

Let p be the probability that A will win the match; then $100p$ will be the value of A's chance. Denote by p_1, p_2, p_3 the respective probabilities that A will win the third and the two preceding games, the fourth and two of the preceding games, the fifth and two of the preceding games; then will

$$p = p_1 + 3p_2 + 6p_3;$$

for A can evidently win two of three games in three ways, and two of four games in six ways. Now

$$p_1 = (\tfrac{8}{11})^3,$$
$$p_2 = (\tfrac{8}{11})^3 (\tfrac{3}{11}),$$
$$p_3 = (\tfrac{8}{11})^3 (\tfrac{3}{11})^2;$$

therefore $$p = p_1 + 3p_2 + 6p_3 = \tfrac{140288}{161051},$$
and $100p$, the value of A's chance, is

$$\$14028800 \div 161051 = \$87.10 +. \qquad [\textit{A. B. Evans.}]$$

68

FROM a point P in the plane of a parabola normals are drawn to the curve. Find the locus of P when the feet of the normals are the corners of a right triangle. [*W. M. Thornton.*]

SOLUTION.

Let $y^2 = 4ax$ be the equation of the parabola, x_1, y_1; x_2, y_2; x_3, y_3 be the co-ordinates of the feet of the normals, and x, y the co-ordinates of P. Then

$$\frac{y_2 - y_1}{x_2 - x_1} \cdot \frac{y_2 - y_3}{x_2 - x_3} = -1,$$

or
$$\frac{y_2 - y_1}{y_2^2 - y_1^2} \cdot \frac{y_2 - y_3}{y_2^2 - y_3^2} = -\frac{1}{16a^2},$$

or
$$(y_1 + y_2)(y_2 + y_3) = -16a^2. \qquad (1)$$

Also y_1, y_2, and y_3 are the roots of the cubic in y',

$$(y - y') + \frac{y'}{2a}\left(x - \frac{y'^2}{4a}\right) = 0,$$

or
$$y'^3 + (8a^2 - 4ax)y' - 8a^2y = 0;$$

$$\therefore\ y_1 + y_2 + y_3 = 0, \qquad (2)$$

$$y_1y_2 + y_3(y_1 + y_2) = 8a^2 - 4ax, \qquad (3)$$

$$y_1y_2y_3 = 8a^2y. \qquad (4)$$

From (1), (2), (3), and (4),

$$y_1(y_1 + y_2) = 16a^2, \qquad (5)$$

$$y_1y_2 - (y_1 + y_2)^2 = 8a^2 - 4ax, \qquad (6)$$

$$y_1y_2(y_1 + y_2) = -8a^2y. \qquad (7)$$

From (5) and (7), $\quad y_2 = -\frac{1}{2}y. \qquad (8)$

From (5) and (6), $\quad y_2^2 = 4ax - 24a^2. \qquad (9)$

From (8) and (9) we have $\quad y^2 = 16a(x - 6a).$

Therefore the required locus is a parabola.

REMARK.—The locus passes through the points of intersection of the original parabola and its evolute, and touches the latter at the same points.

[*R. H. Graves.*]

69

A STRAIGHT LINE is tangent to the parabola $8x = y^2$ and to the circle $x^2 + y^2 = 4$. Required the angle which this line makes with the x-axis.

[*O. Root.*]

SOLUTION.

If m denotes the tangent of the required angle, the line will have for its equation

$$y = mx + \frac{2}{m},$$

or
$$y = mx + 2\sqrt{(1 + m^2)}.$$

That these may be identical we must have

$$m^4 + m^2 = 1;$$

$$\therefore\ m^2 = \tfrac{1}{2}(\sqrt{5} - 1).$$

There are accordingly two solutions, represented by the equations

$$y = \pm \sqrt{[\tfrac{1}{2}(\sqrt{5} - 1)]} \cdot [x + \sqrt{5} + 1],$$

and the required angle is

$$\pm \tan^{-1} \sqrt{[\tfrac{1}{2}(\sqrt{5} - 1)]},$$

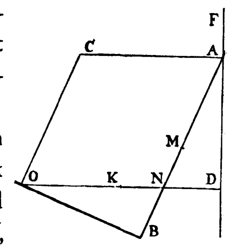
or $\pm 38^\circ 10'$ nearly. [*T. U. Taylor.*]

70

A RIGHT ANGLE moves so that a given point in one side, distant c from the vertex, lies in a fixed axis, while the other side passes through a fixed point, distant c from this axis. Find the locus of the instantaneous centres or centrodes of the motion. [*R. H. Graves.*]

SOLUTION.

Let B be the moving right angle, A the given point lying in the fixed axis DF, while O is a fixed point through which one side constantly passes. Also let $AB = OD$. Draw AC and OC at right angles to AD and OB. Their intersection C is the instantaneous centre.



Evidently $ON = AN$; $\therefore$ $CO = CA$; $\therefore$ locus of C in the fixed plane is a parabola whose focus is O and directrix is FD. Also, in the moving plane, the locus of C is an equal parabola whose focus is A and directrix is OB. M and K, the middle points of AB and OD and the vertices of the parabolas, coincide when AB coincides with OD.

Newton showed that M generates a cissoid. Also, A moves in a straight line. The cissoid and right line are known to be the loci of the vertex and focus of a parabola rolling on an equal parabola, corresponding points being in contact. [*R. H. Graves.*]

[Prof. Bohannan solves in the same way. Dr. Hendricks refers us to a solution in the *Analyst*, V, 55.]

71

A LINE of unit length is bent to the arc of a circle such that the area of the segment it determines is a maximum. Find the radius of the circle and the form and area of the segment. [*O. Root, Jr.*]

SOLUTION.

Assuming that the unit length is not greater than one circumference, the area of the segment is

$$\frac{r}{2} - \frac{r^2}{2} \sin \frac{1}{r},$$

the angle being expressed in radial measure. For a maximum or minimum value,

$$\cos\frac{1}{2r}\left(\cos\frac{1}{2r} - 2r\sin\frac{1}{2r}\right) = 0,$$

i. e. $\cos\frac{1}{2r} = 0$, or $\tan\frac{1}{2r} = \frac{1}{2r}$.

The second relation is only true for a zero-angle, and need not be considered. $\cos\frac{1}{2r} = 0$ gives, under our assumption,

$$\frac{1}{2r} = \frac{\pi}{2}; \quad \therefore r = \frac{1}{\pi}.$$

The variable factor of the second differential co-efficient of the given function is

$$-2\sin\frac{1}{r} + \frac{2}{r}\cos\frac{1}{r} + \frac{1}{r^2}\sin\frac{1}{r}.$$

Since $1/r = \pi$, this expression is negative; therefore the above value of r corresponds to a maximum.

Thus the segment is a semi-circle, and its area, in terms of the square on the given unit (a semi-circumference), is $r/2$ or $1/2\pi$.

[*R. D. Bohannan.*]

EXERCISES.

79

CONSTRUCT a square; given one vertex and two circles on which the extremities of the opposite diagonal lie. [*R. D. Bohannan.*]

80

IN exercise 4, what is the probability that the random circle exceeds the average circle. [*Artemas Martin.*]

81

FIND the equation to the curve whose ordinates represent the areas of the triangles in exercise 68. [*R. H. Graves.*]

82

THE major axes of two similar and equal concentric ellipses intersect at right angles, and the area common to the two curves is half that of either ellipse. Find the excentricity. [*Ormond Stone.*]